

Towards the Cardy formula for hyperscaling violation black holes

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The aim of this paper is to propose a generalized Cardy formula in the case of three-dimensional hyperscaling violation black holes. We first note that for the hyperscaling violation metrics, the scaling of the entropy in term of the temperature (defined as the effective spatial dimensionality divided by the dynamical exponent) depends explicitly on the gravity theory. Starting from this observation, we first explore the case of quadratic curvature gravity theory for which we derive four classes of asymptotically hyperscaling violation black holes. For each solution, we compute their masses as well as those of their soliton counterparts obtained through a double Wick rotation. Assuming that the partition function has a certain invariance involving the effective spatial dimensionality, a generalized Cardy formula is derived. This latter is shown to correctly reproduce the entropy where the ground state is identified with the soliton. Comparing our formula with the one derived in the standard Einstein gravity case with source, we stress the role played by the effective spatial dimensionality. From this observation, we speculate the general form of the Cardy formula in the case of hyperscaling violation metric for an arbitrary value of the effective spatial dimensionality. Finally, we test the viability of this formula in the case of cubic gravity theory.

I. INTRODUCTION

In the last decade, there are been a real interest in extending the ideas underlying the AdS/CFT correspondence [1] in order to gain a better understanding of strongly condensed matter systems. In condensed matter physics, a quantum phase transition is a transition that occurs between two different phases at zero temperature. At this critical point, the system may enjoy an anisotropic scaling symmetry or even display an hyperscaling violation which is reflected by the fact that the entropy S scales with respect to the temperature T as [2–4]

$$S \sim T^{\frac{d_{\text{eff}}}{z}}, \quad (1)$$

where d_{eff} is the effective spatial dimensionality, and where z is the dynamical exponent responsible of the anisotropy of the system

$$t \mapsto \lambda^z t, \quad \vec{x} \mapsto \lambda \vec{x}. \quad (2)$$

In the non-relativistic version of the AdS/CFT correspondence, the gravity dual metric in the anisotropic case is played by the Lifshitz metric whose line element in D dimensions is given by [5]

$$ds_L^2 = -\frac{r^{2z}}{l^2} dt^2 + \frac{l^2}{r^2} dr^2 + \frac{r^2}{l^2} d\vec{x}^2, \quad (3)$$

where $\vec{x} = (x^1, \dots, x^{D-2})$. For the Lifshitz metric, it is easy to see that the anisotropic transformations (2) together with $r \rightarrow \lambda^{-1} r$ act as an isometry. It is well-known also that, in contrast to the AdS isotropic case $z = 1$, Lifshitz metrics or their black hole extensions are not solutions of the standard Einstein gravity but

instead require the introduction of some extra matter source and/or to consider higher-order gravity theories, see e. g. [6–13]. Moreover, in the Lifshitz case, the effective spatial dimensionality has a fixed value which depends on the dimension as $d_{\text{eff}} = D - 2$.

On the other hand, systems which display an hyperscaling are described by the so-called hyperscaling violation metric whose line elements is conformally related to the Lifshitz metric as

$$ds^2 = \frac{1}{r^{\frac{2\theta}{D-2}}} \left[-r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}^2 \right], \quad (4)$$

where now the transformations (2) together with $r \rightarrow \lambda^{-1} r$ act rather like a conformal transformation, $ds^2 \rightarrow \lambda^{2\theta/(D-2)} ds^2$. Examples of hyperscaling violation black holes are known in the literature, see e. g. [14–19]. As shown below, in the hyperscaling violation case, the effective spatial dimensionality will explicitly involve the hyperscaling violation exponent θ , and this dependence will vary in function of the gravity theory considered. For example, in the case of standard three-dimensional Einstein gravity with source, we have $d_{\text{eff}} = 1 - \theta$, see e. g. [18] while (see below) for quadratic (resp. cubic) gravity theories in three dimensions we will have $d_{\text{eff}} = 1 + \theta$ (resp. $d_{\text{eff}} = 1 + 3\theta$).

The aim of this paper is to speculate the form of a generalized Cardy formula in three dimensions in the case of hyperscaling violation metric under the assumption that the ground state is identified with the soliton. This task has been realized in the Lifshitz case by exploiting an isomorphism between the Lifshitz Lie algebras with dynamical exponent z and z^{-1} in two dimensions, see [20]. In this last reference, the Cardy formula has been derived assuming that the ground state is identified with the soliton which is, in addition, separated from the black hole spectrum by a gap. Note that in Ref. [10], we have tested the viability of the Cardy formula of [20] in some concrete examples of Lifshitz black holes with a source given by a scalar field nonminimally coupled. Just to

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be complete, it is worth mentioning that in the standard three-dimensional AdS gravity supported by scalar fields, the identification of the solitons as ground states turns out to be essential for microscopically counting for the black holes entropy using Cardy formula [21].

From now, one can anticipate that, in contrast with the three-dimensional Lifshitz case where the effective spatial dimensionality always takes the value $d_{\text{eff}} = 1$, our proposal for the Cardy formula in the case of hyperscaling violation metric will instead depend explicitly on the exponent θ , and this dependence is inheriting from the theory considered. In order to guess the form of the possible Cardy formula for hyperscaling violation, we will consider the case of quadratic and cubic curvature gravity theories for which we will derive black hole and their soliton counterpart solutions. The black hole and soliton masses will be computed using the quasilocal formalism introduced in [22, 23] where in the next section we will briefly recall the main lines. Comparing our results obtained in the quadratic and cubic cases to those derived recently in [18] in the case of standard Einstein gravity with source, we will speculate the form of the generalized Cardy formula with a ground state identified with the soliton. This generalized formula which reduces to the one derived in [18] in the standard Einstein case will be shown to depend explicitly on the effective spatial dimensionality, and we will show that this formula correctly reproduces the semiclassical entropy in the different examples exploited in this paper.

The plan of the paper is organized as follows. In the next section, we consider the most general quadratic curvature in three dimensions for which we derive four classes of hyperscaling violation black holes. For each solution, we compute their masses as well as well those of their respective soliton counterparts. We also show that there always exists an election of the coupling constant that ensures the black hole masses to be positive while those of their soliton counterparts turn to be negative; this ensures the existence of a gap in the spectrum of the solutions. In addition, assuming that the theory is invariant under a certain modular transformation whose form depends on the effective spatial dimensionality, we will be able to obtain a Cardy formula. This latter under the assumption that the ground state is identified with the soliton, correctly reproduces the expressions of the Wald entropy for each of our examples. Then, comparing our formula with the one derived recently in the standard Einstein case with source [18], we propose a generalized Cardy formula whose expression depends explicitly on the effective spatial dimensionality. In the last section, we check the validity of this Cardy formula in the case of the most general cubic gravity theory in three dimensions. Finally, the last section is devoted to our conclusions.

II. QUADRATIC CURVATURE GRAVITY THEORY

In three dimensions, we consider the most general quadratic curvature gravity theory given by the following action

$$\begin{aligned} S[g_{\mu\nu}] &= \int d^3x \sqrt{-g} (\beta_1 R^2 + \beta_2 R_{\mu\nu} R^{\mu\nu}) \\ &= \int d^3x \sqrt{-g} \mathcal{L}, \end{aligned} \quad (5)$$

where β_1 and β_2 are two coupling constants. The field equations arising from the variation of the action (5) read

$$\begin{aligned} \beta_2 \square R_{\mu\nu} + \frac{1}{2} (4\beta_1 + \beta_2) g_{\mu\nu} \square R - (2\beta_1 + \beta_2) \nabla_\mu \nabla_\nu R \\ + 2\beta_2 R_{\mu\alpha\nu\beta} R^{\alpha\beta} + 2\beta_1 R R_{\mu\nu} \\ - \frac{1}{2} (\beta_1 R^2 + \beta_2 R_{\alpha\beta} R^{\alpha\beta}) g_{\mu\nu} = 0. \end{aligned} \quad (6)$$

Since we are looking for asymptotically hyperscaling violation black holes, we posit the following ansatz for the metric

$$ds^2 = \frac{1}{r^{2\theta}} \left[-r^{2z} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\varphi^2 \right], \quad (7)$$

where the metric function $f(r)$ possesses at least one root, and satisfies $\lim_{r \rightarrow \infty} f(r) = 1$ in order to reproduce asymptotically the hyperscaling violation metric (4). To simplify the computations, we will consider the following particular form for the metric function

$$f(r) = 1 - \left(\frac{r_h}{r} \right)^\alpha, \quad (8)$$

where α is a constant with $\alpha > 0$, and where r_h denotes the location of the horizon. In fact, there exist more general solutions within the ansatz (7) which involve more than an one integration constant. We will discuss one of these solutions at the end of the section devoted to the discussion.

In addition, since we will be interested on the thermodynamic properties of the solutions, we compute the Wald entropy for the generic solution (7-8) which is given by

$$\begin{aligned} \mathcal{S}_W &= 8\pi^2 \alpha (r_h)^{1+\theta} \left[(8\theta - 6z + 2\alpha - 4) \beta_1 \right. \\ &\quad \left. + (3\theta - 3z - 1 + \alpha) \beta_2 \right], \end{aligned} \quad (9)$$

while the Hawking temperature takes the form

$$T = \frac{\alpha (r_h)^z}{4\pi}. \quad (10)$$

It is important to stress from now that the entropy scales with the temperature as

$$\mathcal{S}_W \sim T^{\frac{1+\theta}{z}}, \quad (11)$$

which means that the effective spatial dimensionality d_{eff} as defined in (1) is given by

$$d_{\text{eff}} = 1 + \theta. \quad (12)$$

This is due to the fact that we are considering quadratic curvature gravity theories. Indeed, in the case of standard Einstein-Hilbert gravity supplemented by a source given by the (Maxwell)-dilaton Lagrangian [14, 15, 18], the effective spatial dimensionality is instead $d_{\text{eff}} = 1 - \theta$. From now, one can already conclude to the non-universal character of the scaling of the entropy in term of the temperature in the case of hyperscaling violation metric. In other words, this means that the effective spatial dimensionality will depend on the theory considered. This is in contrast with the Lifshitz case $\theta = 0$ where, independently of the theory, the entropy always scales as $S_W \sim T^{\frac{1}{z}}$, see [20] in the general context and [7], [10–12] for concrete examples. This remark will have its importance when speculating the form of the generalized Cardy in the case of hyperscaling violation metric.

In order to compute the mass of our black hole and soliton solutions, we opt for the quasilocal formalism presented in [22, 23] whose main result lies in the relation established between the off-shell ADT potential $Q_{\text{ADT}}^{\mu\nu}$ and the off-shell Noether potential $K^{\mu\nu}$ in the form

$$\sqrt{-g} Q_{\text{ADT}}^{\mu\nu} = \frac{1}{2} \delta K^{\mu\nu} - \xi^{[\mu} \Theta^{\nu]}, \quad (13)$$

where $\xi^\mu \partial_\mu$ denotes the Killing vector field which in our case is ∂_t , and Θ^μ represents a surface term arising from the variation of the action. More precisely, the expressions appearing in (13) are given by

$$\Theta^\mu = 2\sqrt{-g} \left[P^{\mu(\alpha\beta)\gamma} \nabla_\gamma \delta g_{\alpha\beta} - \delta g_{\alpha\beta} \nabla_\gamma P^{\mu(\alpha\beta)\gamma} \right] \quad (14)$$

$$K^{\mu\nu} = \sqrt{-g} \left[2P^{\mu\nu\rho\sigma} \nabla_\rho \xi_\sigma - 4\xi_\sigma \nabla_\rho P^{\mu\nu\rho\sigma} \right], \quad (15)$$

with $P^{\mu\nu\rho\sigma} = \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}$, where \mathcal{L} is the Lagrangian defined in (5). Since the ansatz metric (7-8) depends continuously on the integration constant r_h^α , one can define the conserved charge associated to the Killing field ∂_t (which corresponds to the mass) in the interior region and not in the asymptotic region by introducing a parameter s with range $s \in [0, 1]$ as $s r_h^\alpha$. The advantage of this re-definition lies in the fact that it allows to interpolate between the free parameter solution $s = 0$ and the solution with $s = 1$. In doing so, the quasi-local charge is defined as [22, 23]

$$\mathcal{M}_{\text{bh}}(\xi) = \int_{\mathcal{B}} dx_{\mu\nu} \left(\Delta K^{\mu\nu}(\xi) - 2\xi^{[\mu} \int_0^1 ds \Theta^{\nu]}(\xi|s) \right), \quad (16)$$

where $\Delta K^{\mu\nu}(\xi) \equiv K_{s=1}^{\mu\nu}(\xi) - K_{s=0}^{\mu\nu}(\xi)$ denotes the variation of the Noether potential from the vacuum solution, and $dx_{\mu\nu}$ represents the integration over the compact co-dimension two-subspace. For the generic solution (7-8),

after a tedious but straightforward computation, this last expression becomes parameterized as

$$\mathcal{M}_{\text{bh}} = 2\pi \left[(r_h)^\alpha \Psi_1 r^{1+\theta+z-\alpha} + (r_h)^{2\alpha} \Psi_2 r^{1+\theta+z-2\alpha} \right] \quad (17)$$

where Ψ_1 and Ψ_2 are constants given by

$$\begin{aligned} \Psi_1 &= \left[4\alpha^3 + (-8 + 4\theta - 8z)\alpha^2 + (-40\theta^2 + 8\theta + 8 - 4z^2 + 36z\theta)\alpha + 4(2z - 1 - 7\theta)(z^2 - 2z\theta + z + 1 + \theta^2 - 2\theta) \right] \beta_1 \\ &\quad + \left[2\alpha^3 + (-4z - 3 + \theta)\alpha^2 + (-15\theta^2 + 15z\theta + 3z + 2\theta - 2z^2 + 1)\alpha - 2z^2 - 18z^2\theta - 6\theta - 2 - 12z\theta + 4z^3 + 24z\theta^2 - 10\theta^3 + 18\theta^2 + 4z \right] \beta_2, \\ \Psi_2 &= \left[-2\alpha^3 + (4z + 5 - \theta)\alpha^2 + (24\theta^2 - 2\theta - 21z\theta - 6 + 2z^2 - 3z)\alpha - 2(2z - 1 - 7\theta)(z^2 - 2z\theta + z + 1 + \theta^2 - 2\theta) \right] \beta_1 \\ &\quad + \left[-\alpha^3 + (2 + 2z)\alpha^2 + (-9z\theta + z^2 - 3z - 1 + 9\theta^2)\alpha + 9z^2\theta + 1 + 3\theta + 6z\theta + z^2 - 2z - 12z\theta^2 - 2z^3 - 9\theta^2 + 5\theta^3 \right] \beta_2. \end{aligned}$$

Since this expression of the mass must be independent of the radial coordinate r , the non-zero black hole solutions, as shown below, will automatically satisfy $\alpha = 1 + \theta + z$ with $\Psi_2 = 0$ or $\alpha = (1 + \theta + z)/2$ with $\Psi_1 = 0$.

Our aim being to derive a generalized version of the Cardy formula for the hyperscaling violation metric where the ground state is played by the soliton, we note that for a black hole solution of the form (7-8), its soliton counterpart obtained through a double Wick rotation will have the following generic form

$$ds^2 = \frac{1}{r^{2\theta}} \left[-r^2 dt^2 + \frac{dr^2}{r^2 \tilde{f}(r)} + r^{2z} \tilde{f}(r) d\varphi^2 \right], \quad (18)$$

$$\tilde{f}(r) = 1 - \left(\frac{\tilde{r}_h}{r} \right)^\alpha,$$

where we have defined

$$\tilde{r}_h = \left(\frac{2}{\alpha} \right)^{\frac{1}{z}}.$$

Along the same lines as before, the quasilocal mass for the soliton (18) can schematically be written as

$$\mathcal{M}_{\text{sol}} = 2\pi \left[(\tilde{r}_h)^\alpha \Phi_1 r^{1+\theta+z-\alpha} + (\tilde{r}_h)^{2\alpha} \Phi_2 r^{1+\theta+z-2\alpha} \right] \quad (19)$$

where Φ_1 and Φ_2 read

$$\begin{aligned}\Phi_1 = & \left[4\alpha^3 + (-12z - 4 + 4\theta)\alpha^2 + (4 + 12z^2 + 12\theta z \right. \\ & + 16\theta - 36\theta^2)\alpha - 4(2z - 3 + 7\theta)(z^2 - 2\theta z + z \\ & + 1 - 2\theta + \theta^2) \left. \right] \beta_1 + \left[\alpha^3 + (2\theta - 3z - 2)\alpha^2 + (6\theta \right. \\ & + 3\theta z + 4z^2 - 13\theta^2 + 3 - z)\alpha + 6 - 4z^3 - 12\theta z \\ & - 22\theta + 6z^2 - 10\theta^3 - 4z - 2z^2\theta + 16\theta^2 z \\ & + 26\theta^2 \left. \right] \beta_2,\end{aligned}$$

$$\begin{aligned}\Phi_2 = & \left[-4\alpha^3 + (14z + 5 - 9\theta)\alpha^2 + (-14z^2 - 10\theta \right. \\ & - 2 + 20\theta^2 + 3\theta z - 3z)\alpha + 2(2z - 3 + 7\theta)(z^2 \\ & - 2\theta z + z + 1 - 2\theta + \theta^2) \left. \right] \beta_1 + \left[-\alpha^3 + (-4\theta \right. \\ & + 4z + 2)\alpha^2 + (-3 - 5z^2 + z + 3\theta z + 7\theta^2 - 4\theta)\alpha \\ & - 3 + 2z^3 + 6\theta z + 11\theta - 3z^2 + 5\theta^3 + 2z + z^2\theta \\ & - 8\theta^2 z - 13\theta^2 \left. \right] \beta_2.\end{aligned}$$

A. Four classes of black hole solutions and their soliton counterparts

In what follows, we present four classes of black hole solutions of the field equations (6) within the ansatz given by (7-8). For each solution, we compute the mass (16) as well as the mass of their soliton counterpart (18) through the generic expression given by (19). We check that the first law of black hole thermodynamics

$$d\mathcal{M}_{\text{bh}} = T d\mathcal{S}_W, \quad (20)$$

is valid in these four cases.

The first family of solutions is obtained for an hyper-scaling violation exponent and dynamical exponent that take the values $\theta = 2$ and $z = 1$, and the line element is given by

$$ds^2 = \frac{1}{r^4} \left[-r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\varphi^2 \right], \quad (21)$$

$$f(r) = 1 - \left(\frac{r_h}{r} \right)^4,$$

provided that the coupling constants β_1 and β_2 are tied as

$$\beta_1 = -\frac{5}{13} \beta_2. \quad (22)$$

In this case, the Wald entropy $\mathcal{S}_W(9)$, the Hawking temperature T (10) and the black hole mass \mathcal{M}_{bh} (17) are given by

$$\begin{aligned}\mathcal{S}_W &= \frac{256}{13} \pi^2 \beta_2 (r_h)^3, & T &= \frac{r_h}{\pi}, \\ \mathcal{M}_{\text{bh}} &= \frac{192}{13} \pi \beta_2 (r_h)^4,\end{aligned} \quad (23)$$

and it is a matter of check to see that the first law (20) is satisfied. The corresponding soliton whose line element (18) written in terms of the "regular" coordinates

$$r = \frac{1}{2\sqrt{\sin\left(\frac{\rho}{2}\right)}},$$

reads

$$ds^2 = -4 \sin\left(\frac{\rho}{2}\right) dt^2 + d\rho^2 + 4 \sin\left(\frac{\rho}{2}\right) \cos^2\left(\frac{\rho}{2}\right) d\varphi^2 \quad (24)$$

has a mass (19) given by

$$\mathcal{M}_{\text{sol}} = -\frac{4}{13} \beta_2 \pi. \quad (25)$$

The second and third solution are identified as black string solutions since the hyperscaling dynamical exponent $\theta = 1$. The first family of black string solution exists for $z = 4$, and its line element is

$$ds^2 = -r^6 f(r) dt^2 + \frac{dr^2}{r^4 f(r)} + d\varphi^2, \quad (26)$$

$$f(r) = 1 - \left(\frac{r_h}{r} \right)^6,$$

provided that the coupling constants β_1 and β_2 are tied as

$$\beta_1 = -\frac{1}{3} \beta_2. \quad (27)$$

For this solution, the thermodynamics quantities are given by

$$\begin{aligned}\mathcal{S}_W &= -64 \pi^2 \beta_2 (r_h)^2, & T &= \frac{3(r_h)^4}{2\pi}, \\ \mathcal{M}_{\text{bh}} &= -32 \pi \beta_2 (r_h)^6,\end{aligned} \quad (28)$$

and, as before, it is easy to see that the first law (20) holds. On the other hand, using the expression (19), its soliton counterpart is shown to have a mass

$$\mathcal{M}_{\text{sol}} = \frac{64}{9} \sqrt{3} \pi \beta_2. \quad (29)$$

The other black string solution $\theta = 1$ arises for $z = 1$ and is given by

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{r^4 f(r)} + d\varphi^2, \quad (30)$$

$$f(r) = 1 - \left(\frac{r_h}{r} \right)^3,$$

with β_1 given by (27). For this black string solution, the thermodynamic quantities read

$$\begin{aligned}\mathcal{S}_W &= 16 \pi^2 \beta_2 (r_h)^2, & T &= \frac{3r_h}{4\pi}, \\ \mathcal{M}_{\text{bh}} &= 8 \pi \beta_2 (r_h)^3, & \mathcal{M}_{\text{sol}} &= -\frac{32}{27} \beta_2 \pi,\end{aligned} \quad (31)$$

and these latter fit perfectly with the first law (20).

The last family of solution corresponds to a Lifshitz black hole (that is $\theta = 0$) with a dynamical exponent $z = 3$,

$$ds^2 = -r^6 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\varphi^2, \quad (32)$$

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^4,$$

provided that the coupling constant β_1 is given by (22). As before, the Wald entropy (9), the Hawking temperature (10) together with the masses of the black hole and its soliton counterpart are given by

$$\mathcal{S}_W = -\frac{256}{13} \pi^2 \beta_2 r_h, \quad T = \frac{(r_h)^3}{\pi},$$

$$\mathcal{M}_{\text{bh}} = -\frac{64}{13} \pi \beta_2 (r_h)^4, \quad \mathcal{M}_{\text{sol}} = \frac{48}{13} 2^{2/3} \beta_2 \pi. \quad (33)$$

In this case again, it is simple to check that the first law (20) is satisfied. It is somehow appealing that in new massive gravity in three dimensions [24], the Lifshitz black hole solution also exist for a dynamical exponent $z = 3$, [7]; this seems to confer a particular status to the value $z = 3$ concerning the Lifshitz black holes in three dimensions for higher-order gravity theories.

We also note that there exist other solutions within the ansatz (7-8), but these latter have a vanishing mass, and hence present a little interest for our main task.

B. Generalized Cardy formula

In the previous sub-section, we have presented four different families of hyperscaling violation black holes with one of them being a Lifshitz black hole solution. In addition to the first first law of thermodynamic (20), the following Smarr formula

$$\mathcal{M}_{\text{bh}} = \frac{1 + \theta}{z + 1 + \theta} T \mathcal{S}_W, \quad (34)$$

also holds for the four classes of solutions derived previously.

Another important feature sharing by these four solutions is that the sign of the coupling constant β_2 can always be fixed such that the black hole mass (resp. the mass of its corresponding soliton) is positive (resp. negative). This ensures the soliton to be separated from the black hole spectrum by a gap.

We are now in position to propose a generalized Cardy formula for the model considered here. We will show that the expression of the semiclassical entropy obtained assuming that the ground state is identified with the soliton coincides with the Wald entropy. In order to achieve this task, we make the assumption that the dual field theory is invariant under the following modular transformation

$$\mathcal{Z}[\beta] = \mathcal{Z} \left[(2\pi)^{1+\frac{1+\theta}{z}} \beta^{-\frac{1+\theta}{z}} \right], \quad (35)$$

where β denotes the inverse of the temperature. This allows to establish a relation between the partition functions at low and high temperature regimes provided that $(1 + \theta)/z > 0$. Note that in the Lifshitz case $\theta = 0$, this modular transformation reduces to the one derived in [20] by exploiting the existence of an isomorphism between the Lifshitz algebras with dynamical exponents z and z^{-1} in $2d$. In the hyperscaling case, there does not exist such an isomorphism. In our opinion, this is not surprising since for the hyperscaling violation metric, in contrast with the Lifshitz case, the effective spatial dimensionality does not have a fixed value but explicitly depends on the theory considered.

For each solution with the appropriate sign of the coupling constant β_2 for which $\mathcal{M}_{\text{bh}} > 0$ and $\mathcal{M}_{\text{sol}} < 0$, the existence of a gap ensures that the partition function at low temperature is dominated by the contribution of the ground state (the soliton),

$$\mathcal{Z}[\beta] \sim \exp(-\beta \mathcal{M}_{\text{sol}}),$$

and with the use of (35), we have that, at the high temperature regime,

$$\mathcal{Z}[\beta] \sim \exp \left(-(2\pi)^{1+\frac{1+\theta}{z}} \beta^{-\frac{1+\theta}{z}} \mathcal{M}_{\text{sol}} \right).$$

Hence, the asymptotic growth number of states at fixed energy \mathcal{M}_{bh} can be obtained from the saddle-point approximation yielding

$$\mathcal{S} = \frac{2\pi}{1 + \theta} \mathcal{M}_{\text{bh}} (1 + \theta + z) \left[-\frac{\mathcal{M}_{\text{sol}}(1 + \theta)}{z \mathcal{M}_{\text{bh}}} \right]^{\frac{z}{z+1+\theta}}. \quad (36)$$

Note that in the Lifshitz case ($\theta = 0$), this formula reduces to the one obtained in [20], and it is a matter of check to see that for the four classes of solutions derived previously the expression of the semiclassical entropy coincides with the Wald entropy, $\mathcal{S} = \mathcal{S}_W$. Another important remark is that, in the case of hyperscaling violation for Einstein gravity with a dilaton source, the Cardy formula obtained in [18] differs from (36) by the change $(1 + \theta) \rightarrow (1 - \theta)$ which precisely corresponds to the change of the effective spatial dimensionality d_{eff} . From this last observation, one can speculate the form of the Cardy formula for a generic effective spatial dimensionality d_{eff} as

$$\mathcal{S} = \frac{2\pi}{d_{\text{eff}}} \mathcal{M}_{\text{bh}} (d_{\text{eff}} + z) \left[-\frac{\mathcal{M}_{\text{sol}} d_{\text{eff}}}{z \mathcal{M}_{\text{bh}}} \right]^{\frac{z}{d_{\text{eff}}+z}}. \quad (37)$$

As before, this formula can be derived assuming that the partition function being invariant under the following modular transformation

$$\mathcal{Z}[\beta] = \mathcal{Z} \left[(2\pi)^{1+\frac{d_{\text{eff}}}{z}} \beta^{-\frac{d_{\text{eff}}}{z}} \right]. \quad (38)$$

In other words, this means that for a hyperscaling violation black hole whose entropy scales as (1), the form of the generalized Cardy formula will be given by (37).

In what follows, we will confirm this guess in a concrete example of cubic gravity where the effective spatial dimensionality will be $d_{\text{eff}} = 1 + 3\theta$. Just to conclude this section, let us mention that a Smarr formula can also be obtained in this generic case from the expression of the entropy (37) together with the use of the first law yielding

$$\mathcal{M}_{\text{bh}} = \frac{d_{\text{eff}}}{z + d_{\text{eff}}} T \mathcal{S}, \quad (39)$$

and generalizing the expression (34) obtained in the quadratic case.

III. EXTENSION TO CUBIC GRAVITY THEORY

In order to explore the viability of the generalized Cardy formula (37), we now consider the case of cubic gravity theory in three dimensions with an action given by [25]

$$\begin{aligned} S[g_{\mu\nu}] &= \int d^3x \sqrt{-g} \left(\sum_{i=1}^3 \gamma_i \mathcal{L}_i \right) \\ &= \int d^3x \sqrt{-g} \mathcal{L}, \end{aligned} \quad (40)$$

where

$$\mathcal{L}_1 = R^3, \quad \mathcal{L}_2 = RR^{ab}R_{ab}, \quad \mathcal{L}_3 = R^{ab}R_{bc}R^c_a.$$

The field equations obtained by varying the action (40) read [26]

$$\sum_{i=1}^3 \gamma_i G_{(i)ab} = 0, \quad (41)$$

where we have defined

$$\begin{aligned} G_{(1)ab} &= 3R^2R_{ab} + 3\nabla_p \nabla_q (g_{ab} g^{pq} R^2 - g_a^p g_b^q R^2) \\ &\quad - \frac{1}{2} g_{ab} \mathcal{L}_1, \\ G_{(2)ab} &= R_{ab} R^{cd} R_{cd} + 2RR^{cd} R_{acbd} \\ &\quad + \nabla_p \nabla_q (g_{ab} g^{pq} R^{cd} R_{cd} + g^{pq} R R_{ab} \\ &\quad - g_a^p g_b^q R^{cd} R_{cd} + g_{ab} R R^{pq} - g_b^p R R_a^q \\ &\quad - g_a^p R R_b^q) - \frac{1}{2} g_{ab} \mathcal{L}_2, \\ G_{(3)ab} &= 3R_{acbd} R^{ec} R_e^d + \frac{3}{2} \nabla_p \nabla_q (g^{pq} R_a^c R_{bc} \\ &\quad + g_{ab} R^{ep} R_e^q - g_b^p R^{qc} R_{ac} - g_a^p R^{qc} R_{bc}) \\ &\quad - \frac{1}{2} g_{ab} \mathcal{L}_3. \end{aligned}$$

For an ansatz metric of the form (7-8), the effective spatial dimensionality (1) will be given by $d_{\text{eff}} = 1 + 3\theta$.

Let us check the viability of the generalized Cardy formula (37) with the following solution of the field equations (41) found for $\theta = 1$ and $z = 6$,

$$\begin{aligned} ds^2 &= -r^{10} f(r) dt^2 + \frac{dr^2}{r^4 f(r)} + d\varphi^2, \\ f(r) &= 1 - \left(\frac{r_h}{r} \right)^{10}, \end{aligned} \quad (42)$$

where the coupling constants are tied as

$$\gamma_1 = -\frac{11}{30} \gamma_2 - \frac{3}{20} \gamma_3. \quad (43)$$

In this case, the thermodynamics quantities computed as before yield

$$\begin{aligned} \mathcal{S}_W &= 2880 \pi^2 (4\gamma_2 + 3\gamma_3) (r_h)^4, \quad T = \frac{5(r_h)^6}{2\pi}, \\ \mathcal{M}_{\text{bh}} &= 2880 \pi (4\gamma_2 + 3\gamma_3) (r_h)^{10}, \end{aligned} \quad (44)$$

and it is simple to check that the first law (20) is satisfied. On the other hand, for the corresponding soliton solution obtained from (18) with $\alpha = 10$, the variation of the Noether potential together with the surface term take the following forms

$$\begin{aligned} \Delta K^{rt} &= -\frac{144}{5} (3\gamma_3 + 4\gamma_2) \sqrt[3]{5}, \\ \int_0^1 ds \Theta^r &= -\frac{288}{5} (3\gamma_3 + 4\gamma_2) \sqrt[3]{5}, \end{aligned} \quad (45)$$

giving a unique value for the mass of the soliton, independent of any integration constant,

$$\mathcal{M}_{\text{sol}} = -\frac{864}{5} \sqrt[3]{5} \pi (3\gamma_3 + 4\gamma_2). \quad (46)$$

As in the quadratic case, there exist a choice of the coupling constants given by $3\gamma_3 + 4\gamma_2 > 0$ which ensures the mass of the black hole to be positive, and at the same time the mass of its soliton counterpart turns to be negative. Finally, it is a matter of check to see that the generalized Cardy formula (37) with $d_{\text{eff}} = 1 + 3\theta = 4$ implies as expected that $\mathcal{S} = \mathcal{S}_W$ for the the cubic solution, and the Smarr formula (39) is also satisfied.

IV. DISCUSSION

The aim of this paper was to guess the possible form for a generalized Cardy formula in the case of hyperscaling violation metric in three dimensions. In order to achieve this task, we have first stressed that, in contrast with the Lifshitz case, the effective spatial dimensionality defined by the scaling of the entropy in term of the temperature (1) does not take a fixed value but clearly depends on the gravity theory considered. For example, in the standard Einstein gravity case $d_{\text{eff}} = 1 - \theta$ while for quadratic corrections we have $d_{\text{eff}} = 1 + \theta$ and in the cubic case,

$d_{\text{eff}} = 1 + 3\theta$. From these observations, we have posited the possible form of the generalized Cardy formula in term of the effective spatial dimensionality d_{eff} . As in the Lifshitz case, the ground state is provided by the soliton which is separated from the black hole spectrum by a gap. We have checked the validity of this formula in different examples in the case of quadratic and cubic gravity theories while in the standard Einstein case our formula reduces to the one proposed in [18]. In all these examples, there exist a choice of the coupling constants that ensures the black hole mass (resp. the soliton mass) to be positive (resp. to be negative) which in turn guarantees the existence of a gap in the spectrum.

Nevertheless, in contrast with the Lifshitz case where the Cardy formula was shown to arise as a consequence of the isomorphism in two dimensions between the Lie algebras with dynamical exponents z and z^{-1} [20], in the present case, we do not have such an argument to justify the modular transformation (38). Hence, a natural extension of this work will be to look for a justification of the duality between the low and high temperature regimes. The exploration of new solutions can also be interesting in order to consolidate the validity of (37). For example, in the quadratic case, there exist most general class of solutions within the ansatz (7) without imposing the form (8) to the metric function. In fact, the first class of string solution (26-27) with $\theta = 1$ and $z = 4$ can be promoted to a two-parametric solution as

$$ds^2 = -r^6 f(r) dt^2 + \frac{dr^2}{r^4 f(r)} + d\varphi^2, \quad (47)$$

$$f(r) = 1 + \frac{a}{r^2} + \frac{a^2}{3r^4} + \frac{b}{r^6},$$

where a and b are two integration constants. Denoting by r_h the location of the horizon

$$(r_h)^2 = \frac{1}{3} \sqrt[3]{(a^3 - 27b)} - \frac{a}{3},$$

the expressions of the Wald entropy and temperature read

$$\mathcal{S}_W = -\frac{64\pi^2}{3} \beta_2 (3r_h^2 + a), \quad T = \frac{(3r_h^2 + a)^2}{6\pi}, \quad (48)$$

while the quasilocal mass which is compatible with the first law involves as well the two integration constants as

$$\mathcal{M}_{\text{bh}} = \frac{32\pi}{27} \beta_2 (27b - a^3) = -\frac{32\pi}{27} \beta_2 (3r_h^2 + a)^3. \quad (49)$$

This dependence of the mass with respect to the two integration constants is similar with the situation occurring with the $z = 1$ AdS black hole solution of new massive gravity [27] where the solution is also two-parametric and, where the two integration constants contribute to the expression of the mass [28]. On the other hand, the soliton counterpart of the solution (47) is shown to have a mass given by (29) and again, it is a simple exercise to check the validity of the Cardy formula (36-37) in this case.

Finally, another interesting task to be realized will be to explore the charged version of the solutions found here in order to propose as well a charged version of the Cardy formula.

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